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NOTE

A Refinement of Carleman's Inequality

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We prove a new sharpening of the inequality

$$\frac{1}{e}\sum_{k=1}^{\infty}\prod_{i=1}^{k} a_i^{1/k} < \sum_{k=1}^{\infty} a_k \qquad (a_k > 0,\ k = 1, 2, ...),$$

which is due to T. Carleman.

In 1923, T. Carleman [4] presented the following inequality: Let a_k $(k = 1, 2, ...)$ be positive real numbers such that $\sum_{k=1}^{\infty} a_k$ is convergent. Then

$$\frac{1}{e}\sum_{k=1}^{\infty} G_k < \sum_{k=1}^{\infty} a_k, \tag{1}$$

where $G_k = \prod_{i=1}^{k} a_i^{1/k}$ denotes the geometric mean of $a_1, ..., a_k$. The constant factor $1/e$ is best possible.

Carleman's inequality has found much attention among several mathematicians, and in many papers different proofs, extensions, and variants of inequality (1) have been provided. We refer to [1, 2, 5–9] and the references given therein.

It is the aim of this note to establish a sharpening of Carleman's inequality which we could not locate in the literature. A basic tool in our proof will be the following refinement of the arithmetic mean–geometric mean inequality which is due to R. Rado.

Lemma. *If* x_k $(k = 1, ..., n)$ *are positive real numbers, then*

$$\frac{1}{n}\Big[\max_{1 \leqslant i \leqslant n} \sqrt{x_i} - \min_{1 \leqslant i \leqslant n} \sqrt{x_i}\Big]^2 \leqslant \frac{1}{n}\sum_{k=1}^{n} x_k - \prod_{k=1}^{n} x_k^{1/n}. \tag{2}$$

A detailed proof of (2) as well as many related results can be found in [3].

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THEOREM. *Let* a_k $(k=1, 2, ...)$ *be positive real numbers such that* $\sum_{k=1}^{\infty} a_k$ *is convergent. Then we have*

$$\frac{1}{e}\sum_{k=1}^{\infty} G_k + \sum_{k=1}^{\infty} \frac{(M_k - m_k)^2}{k(k+1)} < \sum_{k=1}^{\infty} a_k, \tag{3}$$

where

$$M_k = \max_{1 \leqslant i \leqslant k} \sqrt{ia_i} \qquad \text{and} \qquad m_k = \min_{1 \leqslant i \leqslant k} \sqrt{ia_i}.$$

Proof. Let $k \geqslant 1$ be an integer. From the lemma (with $x_k = ka_k$) we obtain

$$\sqrt[k]{k!}\, G_k = \prod_{i=1}^{k} (ia_i)^{1/k} \leqslant \frac{1}{k} \sum_{i=1}^{k} ia_i - \frac{(M_k - m_k)^2}{k}. \tag{4}$$

We multiply both sides of (4) by $1/(k+1)$ and sum from $k=1$ to $k=n$. This yields

$$\begin{aligned}\sum_{k=1}^{n} \frac{\sqrt[k]{k!}}{k+1} G_k + \sum_{k=1}^{n} \frac{(M_k - m_k)^2}{k(k+1)} &\leqslant \sum_{k=1}^{n} \left(\frac{1}{k(k+1)} \sum_{i=1}^{k} ia_i \right) \\ &= \sum_{k=1}^{n} a_k - \frac{1}{n+1} \sum_{k=1}^{n} ka_k \\ &< \sum_{k=1}^{n} a_k.\end{aligned}$$

If we let n tend to ∞, then we have

$$\sum_{k=1}^{\infty} \frac{\sqrt[k]{k!}}{k+1} G_k + \sum_{k=1}^{\infty} \frac{(M_k - m_k)^2}{k(k+1)} \leqslant \sum_{k=1}^{\infty} a_k.$$

Finally, we use the inequality $(1/e) < \sqrt[k]{k!}/(k+1)$ $(k=1, 2, ...)$, to obtain inequality (3). ∎

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